Math 250 Briggs

5.3 Fundamental Theorem of Calculus FTC

Objectives

1) Understand and use area functions

$$A(x) = \int_a^x f(t)\,dt$$

2) See proof of FTC (part 2)

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$

3) Use FTC (part 2) to prove FTC (part 1)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

* 4) Use FTC (part 2) to evaluate definite integrals

* 5) Use FTC (part 1) to find derivatives of area functions.

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① Let $f(t) = 3t - 3$ and consider the two area functions

$$A(x) = \int_1^x f(t)\,dt$$

$$F(x) = \int_4^x f(t)\,dt$$

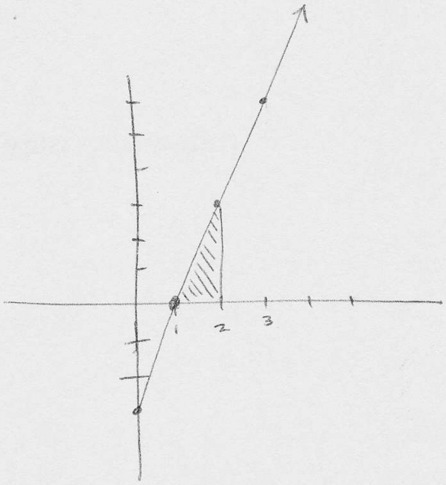
a) Evaluate $A(2)$ and $A(3)$. Then use geometry to find an expression for $A(x)$, $x \geq 1$.

$$A(2) = \int_1^2 3t - 3\,dt$$

$= $ area of Δ

$= \frac{1}{2}BH$

$= \frac{1}{2}(1)(3) = \boxed{\frac{3}{2}}$



$$A(3) = \int_1^3 3t - 3\,dt$$

$= $ area of $\Delta = \frac{1}{2}BH = \frac{1}{2}(2)(6) = \boxed{6}$

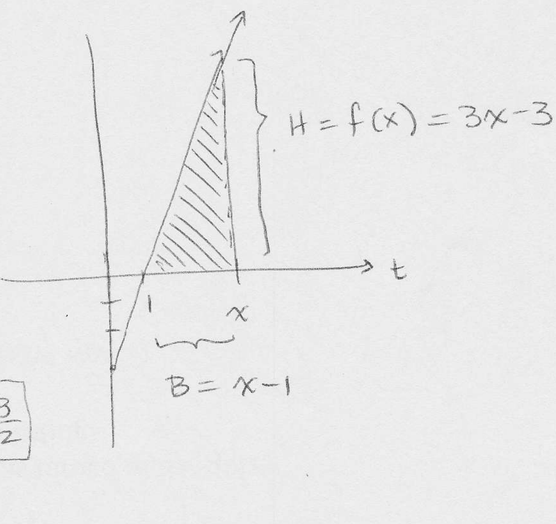
$$A(x) = \int_1^x 3t - 3\,dt$$

$= \frac{1}{2}BH$

$= \frac{1}{2}(x-1)(3x-3)$

$= \frac{3}{2}(x-1)(x-1)$

$= \boxed{\frac{3}{2}(x-1)^2}$ or $\boxed{\frac{3}{2}x^2 - 3x + \frac{3}{2}}$



b) Evaluate $F(5)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, $x \geq 4$.

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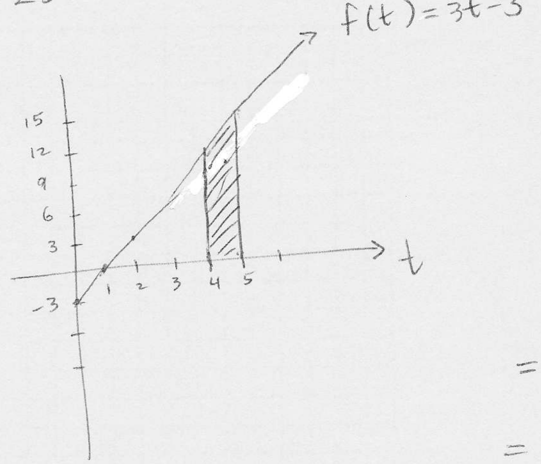
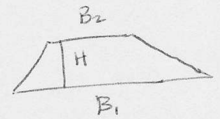
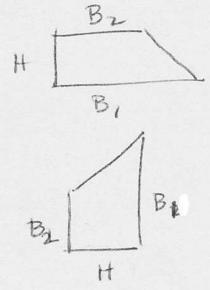


$$F(5) = \int_4^5 f(t)\,dt$$

$= \text{area of trapezoid}$



$A = \frac{1}{2}(B_1+B_2)H$



$= \frac{1}{2}(B_1 + B_2)\cdot H$

$= \frac{1}{2}\big(f(4) + f(5)\big)(5-4)$

$= \frac{1}{2}(9 + 12)\cdot 1$

$= \boxed{\frac{21}{2}}$

$$F(6) = \int_4^6 f(t)\,dt = \text{area of trapezoid} = \frac{1}{2}(B_1+B_2)\cdot H$$

$= \frac{1}{2}\big(f(4)+f(6)\big)\cdot(6-4)$

$= \frac{1}{2}(9+15)\cdot 2$

$= \boxed{24}$

$$F(x) = \int_4^x 3t-3\,dt = \frac{1}{2}(B_1+B_2)\cdot H$$

$= \frac{1}{2}\big(f(4)+f(x)\big)\cdot(x-4)$

$= \frac{1}{2}(9+3x-3)(x-4)$

$= \frac{1}{2}(3x+6)(x-4)$

$= \boxed{\frac{3}{2}(x+2)(x-4)}$ or $\frac{3}{2}(x^2-2x-8)$

$= \boxed{\frac{3}{2}x^2 - 3x - 12}$

c) Show that $F(x) - A(x)$ is a constant.

Show that $F'(x) = A'(x) = f(x)$.

$F(x) - A(x) = \left(\frac{3}{2}x^2 - 3x - 12\right) - \left(\frac{3}{2}x^2 - 3x + \frac{3}{2}\right) = -12 - \frac{3}{2} = -\frac{27}{2}$ ✓

$F'(x) = \frac{d}{dx}\left(\frac{3}{2}x^2 - 3x - 12\right) = 3x - 3 = f(x)$ ✓

$A'(x) = \frac{d}{dx}\left(\frac{3}{2}x^2 - 3x + \frac{3}{2}\right) = 3x - 3 = f(x)$ ✓

The Fundamental Theorem of Calculus (FTC -part 2)

If f is continuous on $[a,b]$

and F is an antiderivative of f on $[a,b]$

Then

$$\underbrace{\int_a^b f(x)\,dx}_{\text{net signed area of plane region}} = \underbrace{F(b) - F(a)}_{\text{evaluate a different but related function twice}}$$

THIS IS AN AMAZING RESULT.

To prove the FTC, we need to remember the MVT

Mean Value Theorem (from chapter)

If f is continuous on $[a,b]$

and f is differentiable on (a,b)

then there exists at least one point c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

i.e. slope of tangent at $x=c$ = slope of secant through $(a, f(a))$ and $(b, f(b))$

The existence of c is what we'll use in the proof.

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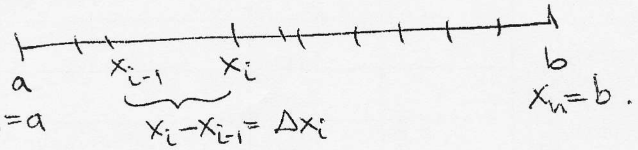
But we're going to use the MVT for the function F on each subinterval of a partion



<u>MVT</u> in notation we'll be using:

If F is continuous on $[x_{i-1}, x_i]$ — ② F diff $\Rightarrow$ F cont

and F is differentiable on (x_{i-1}, x_i) — ① $F'(x)=f(x)$, so F diff.

Then there exists at least one value $x = c_i$

$$x_{i-1} < c_i < x_i$$

such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

← There's a c_i for every single subinterval.

Further notes:

1) What does it mean for F to be an antiderivative of f?

$$F'(x) = f(x).$$

2) $x_i - x_{i-1}$ = width of the i^{th} subinterval $= \Delta x_i$.

So MVT + these notes gives:

$$F'(c_i) = f(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

This is powerful! It lets us take an expression about F and replace it with an expression about f.

Recall also, Defn of Definite Integral

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

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To prove the FTC, we're going to start with

$$F(b) - F(a),$$

do a lot of manipulation, and end up with

$$\int_a^b f(x)\,dx.$$

Proof:

Let Δ be a partition of $[a,b]$ such that

$$\overset{a}{=x_0} < x_1 < \dots < x_i < \dots \overset{b}{=x_n}$$

with $F(x_0) = F(a)$ and $F(x_n) = F(b)$.

$$F(b) - F(a) = F(x_n) - F(x_0)$$

$$= \underbrace{F(x_n) - F(x_{n-1})} + \underbrace{F(x_{n-1}) - F(x_{n-2})} + \dots$$

$$\dots + \underbrace{F(x_2) - F(x_1)} + \underbrace{F(x_1) - F(x_0)}$$

← We added $F(x_i)$ and subtracted $F(x_i)$ for each i from 1 to $n-1$. This creates pairs, $F(x_i) - F(x_{i-1})$.

$$= \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$ ← Rewrite with sigma notation

$$= \sum_{i=1}^{n} \frac{F(x_i) - F(x_{i-1})}{\Delta x_i} \cdot \Delta x_i$$ ← Mult and divide by Δx_i

By the MVT $$= \sum_{i=1}^{n} F'(c_i) \cdot \Delta x_i$$ ← Substitute conclusion from MVT as re-written for F on $[x_{i-1}, x_i]$

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$ ← $F'(x) = f(x)$ means $F'(c_i) = f(c_i)$.

Take $\lim\limits_{\|\Delta x\| \to 0}$ of both sides

$$\lim_{\|\Delta x\| \to 0} \left[F(b) - F(a)\right] = \lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$ ← Taking limit gets the defn of definite integral from 4.3

$$F(b) - F(a) = \int_a^b f(x)\,dx$$ ← $F(b)$ and $F(a)$ are constants, unaffected by the limit.

▨ end of proof.

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If we know the FTC (part 2), written here with dummy variable t.

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

We can use this to take the derivative of an area function ... thus proving the FTC (part 1).

Proof Replace $b = x$,

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

where $F'(x) = f(x)$, (an antiderivative)

Add $F(a)$ to both sides

$$F(a) + \int_a^x f(t)\,dt = F(x)$$

Differentiate WRT x

$$\underbrace{\frac{d}{dx}F(a)} + \frac{d}{dx}\int_a^x f(t)\,dt = F'(x)$$

$F(a)$ is a constant, so its derivative is zero

$F'(x) = f(x)$ because it's an antiderivative

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$\boxtimes$

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Notation used for evaluating the antiderivative

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$$

This means subst $x=b$ and subst $x=a$ then subtract.

To evaluate a definite integral using the FTC:

Step 1: Find the antiderivative F.

Step 2: Evaluate $F(b) - F(a)$.

Step 3: Subtract.

Evaluate the definite integrals.

② $\int_1^2 \sqrt{\frac{2}{x}}\,dx$

$= \int_1^2 \frac{\sqrt{2}}{\sqrt{x}}\,dx$

$= \sqrt{2}\int_1^2 x^{-1/2}\,dx$

$= \sqrt{2}\left[2x^{1/2}\right]\Big|_{x=1}^{x=2}$

$= \sqrt{2}\left[2(2)^{1/2} - 2(1)^{1/2}\right]$

$= \sqrt{2}\left[2\sqrt{2} - 2\right]$

$= \boxed{4 - 2\sqrt{2}}$

*Note: The vertical bar (or bracket) is an abbreviation for "evaluate at $x=2$, evaluate at $x=1$ and subtract $F(2) - F(1)$."
The $x=$ is optional; we could have written:

$$\sqrt{2}\left(2x^{1/2}\right)\Big|_1^2$$

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But what happened to the $+C$ in $F(x)$, you may ask?

(4) $\int_1^2 \sqrt{\frac{2}{x}}\, dx$

$= \int_1^2 \frac{\sqrt{2}}{\sqrt{x}}\, dx$

$= \sqrt{2} \int_1^2 x^{-1/2}\, dx$

Find anti-derivative: $2x^{1/2} + C$

$F(b) = F(2) = 2(2)^{1/2} + C = 2\sqrt{2} + C$

$F(a) = F(1) = 2(1)^{1/2} + C = 2 + C.$

Note: This notation is long and irritating. See below for quicker notation.*

$= \sqrt{2}\left((2\sqrt{2} + C) - (2 + C)\right)$

$= \sqrt{2}\,(2\sqrt{2} + C - 2 - C)$

$= \sqrt{2}\,(2\sqrt{2} - 2)$

$= \boxed{4 - 2\sqrt{2}}$

Note: The $+C-C$ always happens with definite integrals, so we'll leave the $+C$ off from now on, for definite integrals. Indefinite integrals still have $+C$.

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Practice. Evaluate the definite integrals.
Use FTC if possible, geometry if not.

yes (3) $$\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx$$

$$= \int_{-8}^{-1} \left(\frac{x}{2\sqrt[3]{x}} - \frac{x^2}{2\sqrt[3]{x}}\right) dx$$ separate numerator into two terms

$$= \int_{-8}^{-1} \left(\frac{x}{2x^{1/3}} - \frac{x^2}{2x^{1/3}}\right) dx$$ exponent

$$= \int_{-8}^{-1} \left(\frac{1}{2}x^{2/3} - \frac{1}{2}x^{5/3}\right) dx$$ subtract exp $1 - 1/3 = 2/3$, $2 - 1/3 = 5/3$

$$= \left[\frac{1}{2}\cdot\frac{3}{5}x^{5/3} - \frac{1}{2}\cdot\frac{3}{8}x^{8/3}\right]_{x=-8}^{x=-1}$$ integrate $2/3 + 1 = 5/3$ exp, $5/3 + 1 = 8/3$ exp

$$= \left[\frac{3}{10}x^{5/3} - \frac{3}{16}x^{8/3}\right]_{-8}^{-1}$$

$$= \left(\frac{3}{10}(-1)^{5/3} - \frac{3}{16}(-1)^{8/3}\right) - \left(\frac{3}{10}(-8)^{5/3} - \frac{3}{16}(-8)^{8/3}\right)$$ evaluate

$$= \left(\frac{3}{10}\left(\sqrt[3]{-1}\right)^5 - \frac{3}{16}\left(\sqrt[3]{-1}\right)^8\right) - \left(\frac{3}{10}\left(\sqrt[3]{-8}\right)^5 - \frac{3}{16}\left(\sqrt[3]{-8}\right)^8\right)$$ write as radicals

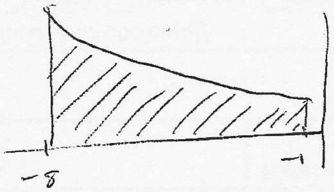
$$= \left[\frac{3}{10}(-1)^5 - \frac{3}{16}(-1)^8\right] - \left[\frac{3}{10}(-2)^5 - \frac{3}{16}(-2)^8\right]$$ evaluate radical

$$= \left[\frac{3}{10}(-1) - \frac{3}{16}(1)\right] - \left[\frac{3}{10}(-32) - \frac{3}{16}(256)\right]$$ evaluate exp

$$= \frac{-3}{10} - \frac{3}{16} + \frac{48}{5} + 48$$

$$= \boxed{\frac{4569}{80}}$$

$$= \boxed{57.1125}$$



$Y_1 = (x - x^2)/(2\cdot x^\wedge(1/3))$

GC check: MATH, 9 fnInt(Y_1, X, -8, -1) = 57.1125 ✓

graphing X

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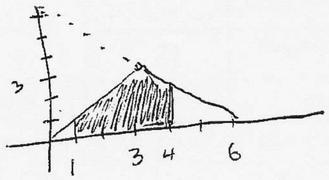
No. 4 $\int_1^4 (3-|x-3|)\,dx$

Graph $f(x) = 3 - |x-3|$

absolute value ✓
- upside down
- shifted up 3
- shifted right 3



Rewrite $f(x)$ as piecewise:

$$f(x) = \begin{cases} x & x \le 3 \\ -x+6 & x > 3 \end{cases}$$

$$= \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

integral property

$$= \int_1^3 x\,dx + \int_3^4 (6-x)\,dx$$

$$= \left[\frac{1}{2}x^2\right]_1^3 + \left[6x - \frac{1}{2}x^2\right]_3^4$$

$$= \frac{1}{2}(3^2) - \frac{1}{2}(1)^2 + \left[6(4) - \frac{1}{2}(4)^2\right] - \left[6(3) - \frac{1}{2}(3)^2\right]$$

$$= \frac{9}{2} - \frac{1}{2} + [24 - 8] - \left[18 - \frac{9}{2}\right]$$

$$= \boxed{6.5} = \boxed{\frac{13}{2}}$$

GC check $\quad y_1 = 3 - \text{abs}(x-3)$ $\quad 6.499994044$

fnInt$(Y_1, X, 1, 4)$

* GC function is using a numerical method like Simpson's Rule to approximate the definite integral. But it is an approximate answer.

Do not use GC as a substitute for doing the exact work! This is a calculus class, not a calculator class!

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(5) $\displaystyle\int_{-3}^{3} v^{1/3}\,dv$

$$= \left[\frac{3}{4} v^{4/3}\right]_{-3}^{3}$$

$$= \frac{3}{4}\left(3^{4/3} - (-3)^{4/3}\right)$$

$$= \frac{3}{4}\left(\left(\sqrt[3]{3}\right)^4 - \left(\sqrt[3]{-3}\right)^4\right)$$

$$= \frac{3}{4}\left(\left(\sqrt[3]{3}\right)^4 - \left(-\sqrt[3]{3}\right)^4\right)$$

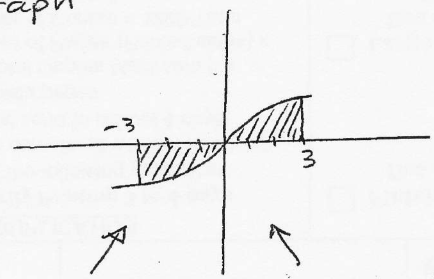
$$= \frac{3}{4}\left(\left(\sqrt[3]{3}\right)^4 - \left(\sqrt[3]{3}\right)^4\right)$$

$$= \frac{3}{4}(0)$$

$$= \boxed{0}$$

GC check fnInt$(Y_1, X, -3, 3) = 0$.
$Y_1 = x\hat{}(1/3)$

Graph



area left of $x=0$ is below x-axis = negative

area right of $x=0$ is above x-axis = positive

Graph is symmetric about the origin, so sizes of shaded areas are equal, so they add to zero.

$y = \sqrt[3]{x}$

$-y = \sqrt[3]{-x}$ subst $(-x,-y)$

$-y = -\sqrt[3]{x}$ simplify

$y = \sqrt[3]{x}$ ✓ div by -1.

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Dummy variables:

If (5) were $\int_{-3}^{3} x^{1/3}\,dx$, does the answer change?

No.

$$\int_{-3}^{3} v^{1/3}\,dv = \int_{-3}^{3} x^{1/3}\,dx = \int_{-3}^{3} \theta^{1/3}\,d\theta = \int_{-3}^{3} q^{1/3}\,dq = 0.$$

The value of the definite integral is independent of the variable used.
The variable has no information, so it's called a "dummy" variable.

Another way of seeing

Using area functions and their derivatives

Find the function $A(x)$ defined by the definite integral.

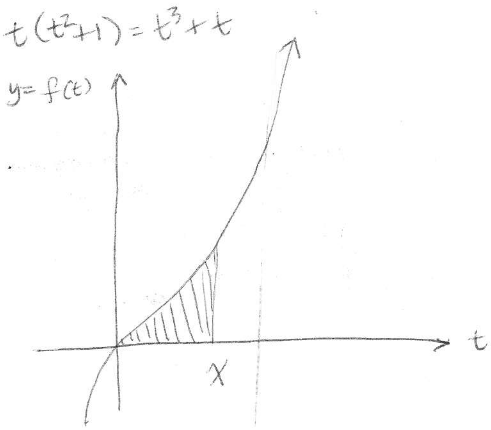
x ⑥ $A(x) = \int_0^x t(t^2+1)\,dt$

$= \int_0^x (t^3+t)\,dt$

$= \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x$

$= \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) - \left(\frac{1}{4}(0)^4 + \frac{1}{2}(0)^2\right)$

$\boxed{A(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2}$



Does the variable of integration matter?

x ⑦ $A(x) = \int_0^x u(u^2+1)\,du$

skip

$= \int_0^x (u^3+u)\,du$

$= \left[\frac{1}{4}u^4 + \frac{1}{2}u^2\right]_{u=0}^{u=x}$

$= \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) - \left(\frac{1}{4}(0)^4 + \frac{1}{2}(0)^2\right)$

$\boxed{A(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2}$ ← same result as ①, so no, the variable used in the integral does not matter. "dummy variable" The variable used as the limit of integration matters.

x ⑧ Find $A'(x)$ when $A(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2$

$\boxed{A'(x) = x^3 + x}$

Notice: $'(x) = \frac{d}{dx}\left[\int_0^x (t^3+t)\,dt\right] = x^3 + x$

FTC (basic): $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ Differentiation + Integration undo each other

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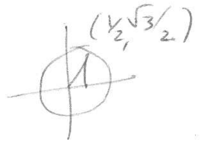
x (9) Find the function defined by the definite integral.

$$K(x) = \int_{\pi/3}^{x} \sec t \tan t \, dt$$

$$= \Big[\sec t\Big]_{t=\pi/3}^{t=x}$$



$$= \sec x - \sec\frac{\pi}{3}$$

$$\boxed{K(x) = \sec x - 2}$$

x (10) Differentiate $K(x) = \sec x - 2$

$$\boxed{K'(x) = \sec x \tan x}$$

Again! $\frac{d}{dx}\left[\int_{\pi/3}^{x} \sec t \, \cancel{\tan} \tan t \, dt\right] = \sec x \tan x.$

$$\boxed{\therefore \frac{d}{dx}\left[\int_{a}^{x} f(t)\, dt\right] = f(x) \qquad \text{Fundamental Theorem of Calculus.}}$$

x (11) Integrate

$$K(x) = \int_{6}^{x^2+1} \sqrt{t}\, dt$$

$$= \left[\frac{2}{3} t^{3/2}\right]_{t=6}^{t=x^2+1}$$

$$\boxed{K(x) = \frac{2}{3}(x^2+1)^{3/2} - \frac{2}{3}(6)^{3/2}}$$

x (12) Differentiate $K(x) = \frac{2}{3}(x^2+1)^{3/2} - \frac{2}{3}(6)^{3/2}$

$$K'(x) = (x^2+1)^{1/2} \cdot 2x - 0$$

$$\boxed{K'(x) = 2x\,(x^2+1)^{1/2}}$$

What if we want to differentiate:

$$K(x) = \int_6^{x^2+1} \sqrt{t}\, dt$$

This upper limit is a function of x, not just x.

$$\int_a^{h(x)} f(t)\,dt = F(h(x)) - F(a) \quad \leftarrow \text{1st FTC}$$

So

$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = F'(h(x))\cdot h'(x) - 0$$

← $F(a)$ is a constant.

← $F(h(x))$ is a chain rule.

$$\boxed{\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x)}$$

2nd FTC – version 2 (not in book, but needed for HW)

X 13 Find $\displaystyle \frac{d}{dx}\int_6^{x^2+1} \sqrt{t}\, dt$

In this question $\begin{cases} a = 6 \\ h(x) = x^2+1 \\ f(t) = \sqrt{t}. \end{cases}$

To use 2nd FTC, we need $h'(x) = 2x$ and $f(h(x)) = \sqrt{x^2+1}$.

So

$$\frac{d}{dx}\int_6^{x^2+1} \sqrt{t}\, dt = \boxed{\sqrt{x^2+1}\cdot 2x}$$

Use 2nd FTC to find derivative $F'(x)$.

x (14) $F(x) = \int_1^x \frac{t^2}{t^2+1}\, dt$

$$\boxed{F'(x) = \frac{x^2}{x^2+1}}$$

x (15) $F(x) = \int_0^{x^2} \sin\theta^2 \, d\theta$

$F'(x) = \sin(x^2)^2 \cdot 2x$

$$\boxed{F'(x) = 2x \sin(x^4)}$$

x (16) $F(x) = \int_1^{3x+1} \sqrt[4]{t}\, dt$

$F'(x) = \sqrt[4]{3x+1} \cdot 3$

$$\boxed{F'(x) = 3 \cdot \sqrt[4]{3x+1}}$$

(x) → What if there are functions at both limits of integration?

$$K(x) = \int_{g(x)}^{h(x)} f(t)\, dt$$

Let $F(x)$ be an antiderivative of $f(x)$: $F'(x) = f(x)$.

$K(x) = F(h(x)) - F(g(x))$ 1st FTC

Differentiate with chain rule

$K'(x) = F'(h(x)) \cdot h'(x) - F'(g(x)) \cdot g'(x)$. subst defn of anti-derivative $F'(x) = f(x)$.

$K'(x) = f(h(x)) \cdot h(x) - f(g(x)) \cdot g'(x)$.

$$\boxed{\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\, dt\right] = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)}$$

3rd and most general version

2nd FTC (not in book)

x (17) Differentiate

$$G(x) = \int_x^{x+2} (5t-3)\,dt$$

$$\begin{cases} g(x) = x & g'(x) = 1 \\ h(x) = x+2 & h'(x) = 1 \\ f(x) = 5x-3 \end{cases}$$

$$G'(x) = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$
$$= [5(x+2)-3] \cdot 1 - [5x-3] \cdot 1$$
$$= 5x + 10 - 3 - 5x + 3$$
$$\boxed{G'(x) = 10}$$

x (18) Differentiate

$$G(x) = \int_0^{\sin x} \sqrt{t}\,dt \qquad \begin{cases} h(x) = \sin x \\ f(x) = \sqrt{x} \end{cases} \qquad h'(x) = \cos x$$

$$G'(x) = f(h(x)) \cdot h'(x)$$
$$= \sqrt{\sin x} \cdot \cos x$$
$$\boxed{G'(x) = \cos x \sqrt{\sin x}}$$

x (19) Differentiate

$$G(x) = \int_{-x}^{x} t^3\,dt \qquad \begin{array}{ll} f(x) = x^3 & \\ g(x) = -x & g'(x) = -1 \\ h(x) = x & h'(x) = 1 \end{array}$$

$$G'(x) = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$
$$= x^3 \cdot 1 - (-x)^3 \cdot (-1)$$
$$= x^3 + (-x^3)$$
$$= \boxed{0} \qquad (?)$$

Check: $G(x) = \int_{-x}^{x} t^3\,dt = \left[\frac{1}{4}t^4\right]_{-x}^{x} = \frac{1}{4}x^4 - \frac{1}{4}(-x)^4$

$$= \frac{1}{4}x^4 - \frac{1}{4}x^4$$
$$= 0 \checkmark$$

So $G'(x) = 0$ also.

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✓ (20) Find $\frac{d}{dx}\int_{x^2-4}^{x^3+8} \sqrt{t^2+1}\, dt$

$f(t) = \sqrt{t^2+1}$

$g(x) = x^2-4$

$g'(x) = 2x$

$h(x) = x^3+8$

$h'(x) = 3x^2$

$= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$

$= \sqrt{(x^3+8)^2+1}\cdot 3x^2 - \sqrt{(x^2-4)^2+1}\cdot 2x$

$= \sqrt{x^6+16x^3+64+1}\cdot 3x^2 - \sqrt{x^4-8x+16+1}\cdot 2x$

$= \boxed{x\left\{3\sqrt{x^6+16x^3+65} - 2\sqrt{x^2-8x+17}\right\}}$